Name of college :→ S.S. college.

Topic – uniform convergence (Real Analysis)

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1. Problem :→ Construct an Example of a Point-wise Convergent Sequence which is not uniformly Convergent.

Ex:→ Consider the sequence defined on $[0,1]$ by

$$f_n(x) = x^n$$

Here $\lim_{n\to\infty} x^n = 0$ if $0 \le x < 1$

$= 1$ if $x = 1$

~~Therefore the limit function f is defined by~~ above ~~has~~

thus the function $f_n(x)$ having a definite limit for all values of x in $[0,1]$ as $n \to \infty$. is point-wise convergent.

Now we are to see whether the Convergence is uniform.

for this we consider the interval $[0,1]$. Let $\epsilon > 0$ be given. then

$$|f_n(x) - f(x)| < \epsilon \Rightarrow |x^n - 0| < \epsilon$$

$$\Rightarrow x^n < \epsilon$$

$$\Rightarrow \frac{1}{x^n} > \frac{1}{\epsilon}$$

$$\Rightarrow n \log\frac{1}{x} > \log\frac{1}{\epsilon}$$

$$\Rightarrow n > \log 1/\epsilon \quad \text{—— (1)}$$

It is obvious that—

When $x \to 1$, $n \to \infty$

thus it is not possible to find m such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for every } n \geq m$$

and for every x in $0 \leq x \leq 1$

Hence $\{f_n(x)\}$ is not uniformly convergent in $0 \leq x \leq 1$

If however, we consider the interval $[0, K]$ where $0 < K < 1$, then the greatest value of

$$\frac{\log 1/\varepsilon}{\log 1/x} \text{ is } \frac{\log 1/\varepsilon}{\log \frac{1}{K}}$$

So we can take m = an integer just greater than $\frac{\log 1/\varepsilon}{\log \frac{1}{K}}$ thus sequence $\{f_n(x)\}$ is uniformly convergent in $[0, K]$ where $0 < K < 1$

2. Ex:→ Show that the sequence $\{f_n(x)\}$ where $f_n(x) = \frac{nx}{1+n^2x^2}$ is point-wise convergent but is not uniformly convergent in any interval with 0 as an interior point.

Sol^n. First of all, we shall show that the function is point-wise convergent for every value of x.

Take any real No $x = c$.
then we have.

$$f(c) = \lim_{n\to\infty}\{f_n c\}$$

$$= \lim_{n\to\infty} \frac{nc}{1+n^2c^2}$$

$$= \lim_{n\to\infty} \frac{nc}{n^2(1+c^2/n^2)}$$

$$= \lim_{n\to\infty} \frac{c/n}{1+c^2/n^2} = \frac{0}{1} = 0$$

Similarly, $f(x) = 0$
for all real value of x.
Thus the sequence $\{f_n(x)\}$ is point wise convergent.

To test uniform convergence, we see that for a given $\epsilon > 0$ we have

$$\left| f_n(x) - f(x) \right| < \epsilon$$

$$\Rightarrow \left| \frac{nx}{1+n^2x^2} - 0 \right| < \epsilon$$

if $$\frac{n|x|}{1+n^2x^2} < \epsilon$$

if $$n^2x^2\epsilon - n|x| + \epsilon > 0$$

if $$n > \frac{|x| + \sqrt{x^2 - 4x^2\epsilon^2}}{2x^2\epsilon}$$

if $$n > \frac{1 \pm \sqrt{1-4\epsilon^2}}{2|x|\epsilon} \quad \text{——— (1)}$$

we take $m(\epsilon, x) =$ an integer just greater than $\frac{1+\sqrt{1-4\epsilon^2}}{2\epsilon x}$
it is clear from (1) that if $x \to 0$, $n \to \infty$
so that it is not possible to choose m such that
$|f_n(x) - f(x)| < \epsilon$ for every $n \geq m$ and for every $x \geq 0$

8. Show that if $f_n(x) = nxe^{-nx^2}$, the sequence $\{f_n\}$ is pointwise convergent, but not uniformly convergent in $[0,\infty]$

Sol^n – Here $f_n(x) = nxe^{-nx^2}$

$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = 0$ for all $x \in [0,\infty[$

Hence the sequence is point-wise convergent and that the point-wise limit is the function f such that $f=0$ for all $x \in [0,\infty]$

Hence the sequence is point-wise convergent and that the pointwise limit is the function f such that $f=0$ for all $x \in [0,\infty[$

If possible, let the sequence be uniformly convergent in $[0,\infty[$

Let $\varepsilon > 0$ be given

then there exists a +ve integer m such that for all $n \geq m$ and for all $x \geq 0$, we have

$$|f_n(x) - f(x)| = nxe^{-nx^2} < \varepsilon \quad \text{——} (1)$$

Let $x = 1/m$ and let m_0 be any integer greater than m and $e^2\varepsilon^2$.

Then the inequality (1) holds for $x = \frac{1}{\sqrt{m_0}}$ and $n = m_0$

These give $\frac{\sqrt{m_0}}{e} < \varepsilon \Rightarrow m_0 < e^2\varepsilon^2$ so that we arrive at a contradiction.

Thus the convergence is not uniform in $[0,\infty[$

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