

Name of college $\rightarrow$ S.S. college.

Topic - uniform convergence (Real Analysis)

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1. Problem $\rightarrow$ Construct an example of a point-wise convergent sequence which is not uniformly convergent.

Ex $\rightarrow$ Consider the sequence defined on $[0,1]$ by

$$f_n(x) = x^n$$

$$\text{Here } \lim_{n \rightarrow \infty} x^n = 0 \text{ if } 0 \leq x < 1$$

$$= 1 \text{ if } x = 1$$

Therefore, the limit function f is defined by above has

Thus the function $f_n(x)$ having a definite limit for all values of x in $[0,1]$ as $n \rightarrow \infty$ is point-wise convergent.

Now we are to see whether the convergence is uniform.

for this we consider the interval $[0,1]$.

Let $\epsilon > 0$ be given. then

$$|f_n(x) - f(x)| < \epsilon \Rightarrow |x^n - 0| < \epsilon$$

$$\Rightarrow x^n < \epsilon$$

$$\Rightarrow \frac{1}{x^n} > \frac{1}{\epsilon}$$

$$\Rightarrow n \log \frac{1}{x} > \log \frac{1}{\epsilon}$$

$$\Rightarrow n > \log \frac{1}{\epsilon} \quad \text{--- (1)}$$

It is obvious that-

When $n \rightarrow 1, n \rightarrow \infty$

Thus it is not possible to find m such that-

$$|f_n(x) - f(x)| \leq \epsilon \text{ for every } n \geq m$$

and for every x in $0 \leq x \leq 1$

Hence $\{f_n(x)\}$ is not-uniformly

Convergent in $0 \leq x \leq 1$

if however, we consider the interval $[0, k]$ where $0 < k < 1$, then the greatest value of

$$\frac{\log 1/\epsilon}{\log 1/x} \text{ is } \frac{\log 1/\epsilon}{\log 1/k}$$

So we can take m = an integer just greater than $\frac{\log 1/\epsilon}{\log 1/k}$

Thus sequence $\{f_n(x)\}$ is uniformly convergent in $[0, k]$ where $0 < k < 1$

2. Ex:- Show that the sequence $\{f_n(x)\}$ where $f_n(x) = \frac{nx}{1+n^2x^2}$ is point-wise

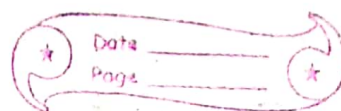
Convergent but is not-uniformly convergent in any interval with 0 as an interior point.

Soln. First- if all, we shall show that the function is point-wise convergent for every value of x .

Take any real no $x = c$.
then we have,

3.

$$f(c) = \lim_{n \rightarrow \infty} f_n(c)$$



$$= \lim_{n \rightarrow \infty} \frac{nc}{1+n^2c^2}$$

$$= \lim_{n \rightarrow \infty} \frac{nc}{n^2(1+c^2/n^2)}$$

$$= \lim_{n \rightarrow \infty} \frac{c/n}{1+c^2/n^2} = \frac{0}{1} = 0$$

Similarly, $f_n \rightarrow 0$

for all real value of x

thus the sequence $\{f_n(x)\}$ is point wise convergent.

To test uniform convergence, we see that for a given $\epsilon > 0$ we have

$$|f_n(x) - f(x)| < \epsilon$$

$$\Rightarrow \left| \frac{nx}{1+n^2x^2} - 0 \right| < \epsilon$$

$$\text{if } \frac{n|x|}{1+n^2x^2} < \epsilon$$

$$\text{if } n^2x^2\epsilon - n|x| + \epsilon > 0$$

$$\text{if } n > \frac{|x| + \sqrt{x^2 - 4x^2\epsilon^2}}{2x^2\epsilon}$$

$$\text{if } n > \frac{1 + \sqrt{1 - 4\epsilon^2}}{2|x|\epsilon} \quad (1)$$

we take $m(\epsilon, x) = \text{an integer just greater than } \frac{1 + \sqrt{1 - 4\epsilon^2}}{2|x|\epsilon}$

it is clear from (1) that if $x \rightarrow 0$, $n \rightarrow \infty$

so that it is not possible to choose m such that

$|f_n(x) - f(x)| < \epsilon$ for every $n \geq m$ and for every $x > 0$

8. Show that if $f_n(x) = nx e^{-nx^2}$ the sequence $\{f_n\}$ is pointwise convergent, but not uniformly convergent in $[0, \infty)$

Soln - Here $f_n(x) = nx e^{-nx^2}$
 $\therefore \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{e^{nx^2}} = 0$ for all $x \in [0, \infty)$

Hence the sequence is point-wise convergent and that the point-wise limit is the function f such that $f=0$ for all $x \in [0, \infty)$

Hence the sequence is point-wise convergent. And that the pointwise limit is the function f such that $f=0$ for all $x \in [0, \infty)$

If possible let the sequence be uniformly convergent in $[0, \infty)$
 let $\epsilon > 0$ be given

then there exists a +ve integer m such that for all $n \geq m$ and for all $x \geq 0$, we have

$$|f_n(x) - f(x)| = nx e^{-nx^2} < \epsilon \quad \text{--- (1)}$$

let $x = \frac{1}{\sqrt{m}}$ and let m_0 be any integer greater

than m and $e^2 \epsilon^2$.

then the inequality (1) holds for $x = \frac{1}{\sqrt{m_0}}$

And $n = m_0$

these give $\frac{1}{\sqrt{m_0}} < \epsilon \Rightarrow m_0 < \frac{1}{\epsilon^2}$ so that

we arrive at a contradiction.

Thus the convergence is not uniform in $[0, \infty)$